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Equation (A) is the general equation of a curve of degree $2n + 2$ having at infinity a point of multiplicity $2n + 1$, whose $2n + 1$ tangents are the line at infinity and $2n$ lines parallel to the x-axis. Such a curve is uniquely determined when c and α are given. The positions of the asymptotes $Y_{2n} = 0$ are independent of α.

It is not difficult to trace the changes in the form of the curve as α varies. It makes $4n + 4$ real intersections with the circle, the extra point having a vectorial angle $-(4n + 3)\alpha$. This extra point always crosses an asymptote at the same time as a vertex crosses the same asymptote on the opposite side of the circle, and the curve then becomes composite, the asymptote being a part of the curve. When no vertex is on an asymptote, the strips into which the asymptotes divide the plane contain two vertices each, one to the right and one to the left, except that the strip containing the extra intersection contains three vertices. This strip also contains (if $\alpha \neq \pm \pi/2$) the only point on the curve where the tangent is parallel to the y-axis. When $\alpha = \pm \pi/2$ (that is, when in equation (1) $b^2 = -a^{11}$), the curve becomes six straight lines.

To obtain similar results for a regular polygon of $4n + 1$ sides (as in Mr. Haldeman's former article on the pentagon) we may start from the equation

$$\sin \{2n\theta - (4n + 1)\alpha\} = -\sin (2n + 1)\theta,$$

and proceed in very much the same way.

An equally simple type of curve (for a polygon of m sides) is $xY_{m-1} = Y_{m-1}$, that is a curve of degree m with $m - 1$ asymptotes parallel to the x-axis and one parallel to the y-axis. To get this, we use, if $m = 4n + 3$, the equation

$$\sin \{(2n + 1)\theta - (4n + 3)\alpha\} = -\sin (2n + 2)\theta;$$

and, if $m = 4n + 1$, the equation

$$\cos \{2n\theta - (4n + 1)\alpha\} = \cos (2n + 1)\theta.$$

III. A GENERALIZATION OF THE PYTHAGOREAN THEOREM.

By J. ROSENBAUM, Bloomfield, Conn.

There is a given angle XOY, and a given number n, and we look for pairs of points, P and Q on OX and OY respectively, such that

$$\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n.$$

When the given angle is a straight angle or a right angle, such a pair of points does not exist unless $n = 1$ or $n = 2$, respectively, and for these values of n any pair of points will satisfy the condition. Conversely, if $n = 1$ or $n = 2$, there is no solution unless the angle is a straight angle or a right angle respectively, and for these angles any pair of points will do. Furthermore, for n such that $0 \leqq n < 1$ there is no solution, because then the condition would require either that PQ be greater than $PO + OQ$, or that 1 be equal to $1 + 1$.

We shall accordingly confine our attention to values of n greater than 1 and different from 2. (Solutions also exist for negative n's but we shall not consider them here.)

It is easy to prove that if P_1, Q_1 is a pair of points satisfying the condition, then all pairs of points P, Q, such that triangles POQ (or QOP) and P_1OQ_1 are similar, will also satisfy the condition. In accordance with the terminology used in algebra, we shall refer to solutions of that kind as dependent. Since we are looking for independent solutions, we can denote PO by 1, and OQ by x.

From the required condition together with the law of cosines, we obtain

$$(x^n + 1)^{2/n} = x^2 + 1 - 2xz, \qquad \text{where} \qquad z = \cos XOY. \tag{1}$$

Solving for z,

$$z = \frac{x^2 + 1 - (x^n + 1)^{2/n}}{2x} = f(x, n). \tag{2}$$

We notice that $f(x, n) = f(1/x, n)$ which means nothing more than this: if in triangle POQ $\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n$, then there is a point Q' on OY such that $\overline{PQ'}^n = \overline{PO}^n + \overline{OQ'}^n$, where $OQ' = (1/OQ)$. The triangles POQ and $Q'OP$ are similar, so that the pairs of points P, Q and P, Q' are not independent.

In what follows we shall endeavor to prove that the converse is also true; i.e. if $f(x_1, n) = f(x_2, n)$, then $x_1 = (1/x_2)$. This will say that if P, Q and P, Q' both satisfy the condition the triangles POQ and $Q'OP$ are similar; i.e. for every given angle excepting for a straight angle or a right angle there is no more than one independent solution. This will be accomplished by proving that for any given $n > 2$, f is a decreasing function of x for all $x > 1$, and that for any given n between 1 and 2, f is an increasing function of x for all $x > 1$.

Differentiating (2) with regard to x,

$$\frac{\partial z}{\partial x} = \frac{(x^2 - 1)(x^n + 1)^{(n-2)/n} - (x^n - 1)}{2x^2(x^n + 1)^{(n-2)/n}}. \tag{3}$$

It is this derivative which we wish to prove is either always positive or always negative for $x > 1$.

Since x is to take on all values greater than 1, and since n is positive, we can make the substitution

$$x = y^{1/n}.$$

The numerator of the right-hand member of (3) now becomes

$$(y^{2/n} - 1)(y + 1)^{(n-2)/n} - (y - 1).$$

This equals $(y + 1)^{-2/n} \cdot \phi(y, n)$, where

$$\phi(y, n) = (y^{2/n} - 1)(y + 1) - (y - 1)(y + 1)^{2/n}. \tag{4}$$

We may disregard the factor $(y + 1)^{-2/n}$ because y is positive, and it is only a question of whether $\partial z/\partial x$ is positive or negative.

Differentiating (4) with regard to n, we have

$$\frac{\partial \phi}{\partial n} = 2n^{-2}(y-1)(y+1)^{2/n} \log (y+1) - 2n^{-2}(y+1)y^{2/n} \log y$$
$$= 2n^{-2}(y+1)(y-1)y[\psi(y+1) - \psi(y)],$$

where $\psi(y) = (y-1)^{-1}y^{(2-n)/n} \log y$.

For $n = 1$, $\psi(y) = (y \log y)/(y-1)$. This is an increasing function of y for all values of $y > 1$, as may be proved by differentiating it and then differentiating the numerator of its derivative.

Therefore $\partial\phi/\partial n > 0$ for $n = 1$.

For $n = 2$, $\psi(y) = (\log y)/(y-1)$; which is found in the same way to be a decreasing function of y for all values of $y > 1$.

Therefore $\partial\phi/\partial n < 0$ for $n = 2$.

If we equate $\partial\phi/\partial n$ to zero, putting $\psi(y+1) = \psi(y)$, and take the logarithms of both sides, we shall obtain a linear equation in n, satisfied by only one value of n for any given value of y. This value of n comes between 1 and 2 and, as $\phi(y, 1) = \phi(y, 2) = 0$, it follows that $\phi(y, n)$ is positive when n lies between 1 and 2, and negative when $n > 2$, and therefore that f is an increasing function of x in the former case and a decreasing function of x in the latter case.

When $x = 1$, $f = 1 - 2^{(2-n)/n}$ and as f represents cos XOY, we have the following results:

(1) *For any given* $n > 2$, *and for a given angle* XOY, *there exists one and only one independent pair of points* P, Q *on* OX *and* OY, *respectively, such that* $\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n$, *provided the angle* XOY *satisfies*

$$\text{arc cosine } (1 - 2^{-(n-2)/n}) \leqq XOY \leqq 90°;$$

otherwise such a pair of points does not exist.

(2) *For* n *between* 1 *and* 2, *and for a given angle* XOY *there exists one and only one pair of points satisfying the condition, provided the angle* XOY *satisfies*

$$\text{arc cosine } (1 - 2^{(2-n)/n}) \geqq XOY \geqq 90°;$$

otherwise there is no pair of points to satisfy the condition.

(3) *In either case this range decreases as* n *approaches* 2, *condensing to a single angle* 90°, *but for* 90° *any pair of points is a solution and there is no limit to the number of independent solutions.*

We can also state now this theorem:

If two triangles have an angle of one equal to an angle of the other, and if in each triangle the n-*th power of the side opposite is equal to the sum of the* n-*th powers of the other two sides, then, for* n *greater than* 1 *and different from* 2, *the triangles are similar.*

IV. NOTE ON TRIGONOMETRIC FUNCTIONS.

By ELIJAH SWIFT, University of Vermont.

Mr. R. S. Underwood's note in this MONTHLY, *1921*, 374 (see also *1922*, 255, 346), concerning the irrationality of certain trigonometric functions, leads me to supplement his statements in one particular.